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Fuzzy rationality and Utility Theory Axioms

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Abstract

Following the results in [7], we present in this paper a generalization of classical utility theory when basic preferences are stated by means of "rational" fuzzy preference relations. Rationality of fuzzy preference relations will be measured according to fuzzy rationality measures previously introduced in [5]. An utility function is introduced by using a "boosting" procedure on the fuzzy preference relations.

1. Introduction

It is clear that in all those frameworks where decisions must be made under probabilistic uncertainty using as well approximate reasoning techniques, a "good "extension of the classical axioms of utility theory which is able to deal with fuzzy preferences and rationality of preferences, could turn out to be very useful in the design phase of an "intelligent software agent". A possible extension of such axioms (as given in [13]) was proposed in [7]. The novelty of the approach lies in the use of general fuzzy rationality measures (as defined by the in [5] for formalizing an axiomatic basis for fuzzy utility theory.

1.1. About rationality

Rationality can be seen as consistency of (degree of) preferences. To each individual one can assign a value of rationality between 0 (absolute irrationality) and 1 (absolute rationality). This value (degree) of rationality can be assigned in many different ways, and each of these different assignments corresponds to a different criterion for measuring the rationality of an individual. We will call these criteria *fuzzy rationality measures*. Fuzzy rationality measures have been introduced and formalized in [2, 4, 5, 6].

2. The axioms of utility theory

We will now introduce a reminder about the axioms of utility theory, as proposed in [13].

Let us assume a set of basic alternatives or states, $A_1, \ldots, A_n$. Such states can be combined to obtain new ones which are basically lotteries. In details, if we have the states $A_1, A_2, \ldots, A_l$ then for any probability value $p_1, p_2, \ldots, p_l$ such that $\sum_{i=1}^{l} p_i = 1$ we obtain the lottery $\begin{pmatrix} p_1 & \cdots & p_l \\ A_1 & \cdots & A_l \end{pmatrix}$, that is to say, the state where with probability p_i we have A_i. The axioms of utility theory which would follow, will not be concerned with utility but with preferences. Preferences (in the crisp case) can be modeled as a binary relation $\mu(A_i, A_j) \in \{0, 1\}$ defined over the set of states. Given any "intelligent agent" we have in particular,

- if $\mu(A_i, A_j) > \mu(A_j, A_i)$ we say that the agent strictly prefers A_i to A_j, i.e. $A_i \succ A_j$;
- if $\mu(A_i, A_j) = \mu(A_j, A_i)$ we say that the agent is indifferent to the two states, i.e. $A_i \sim A_j$; later on we will comment upon indifference as being "positive" or "negative";
- if $\mu(A_i, A_j) \geq \mu(A_j, A_i)$ we say that the agent does not prefer A_j over A_i, i.e. $A_i \succeq A_j$. Such a preference is denoted as "weak" preference.

The agent rationality can be characterized, for example, with the absence of (either strict or weak) preference cycles (see [12]). If the binary relation is complete (i.e. $\mu(A_i, A_j) = 1$ or $\mu(A_j, A_i) = 1$ holds for ev-

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ery pair $A_i, A_j \in X$) weak *acyclity* condition is equivalent to classical crisp transitivity of weak binary relations. Classical crisp binary order preference relations, that is, those relations verifying reflexivity and transitivity, represent ideal examples of consistent crisp preference relations.

The above considerations are formally translated into the first two axioms below. The remaining four axioms are, instead, related to the way the preference relation evolves when we move onto lotteries built from the given states.

(A1) Orderability: given any two states A_i, A_j it must be true

$$(A_i \succ A_j) \vee (A_i \sim A_j) \vee (A_i \prec A_j)$$

(A2) Transitivity: given any three states A_i, A_j, A_h it must be true

$$(A_i \succ A_j) \wedge (A_j \succ A_h) \rightarrow (A_i \succ A_h)$$

(A3) Continuity: this axiom expresses the fact that if $(A_i \succ A_j \succ A_h)$ then there exists a probability value p such that the agent is indifferent between A_j and the lottery where that yields A_i with probability p and A_h with probability $1-p$, that is to say

$$A_i \succ A_j \succ A_h \rightarrow \exists p | \begin{pmatrix} p & 1-p \\ A_i & A_h \end{pmatrix} \sim A_j$$

(A4) Substitutability:

$$A_i \sim A_j \rightarrow \begin{pmatrix} p & 1-p \\ A_i & A_h \end{pmatrix} \sim \begin{pmatrix} p & 1-p \\ A_j & A_h \end{pmatrix}$$

(A5) Monotonicity:

$$A_i \succ A_j \rightarrow$$
$$\{p \geq q \leftrightarrow \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix} \succeq \begin{pmatrix} q & 1-q \\ A_i & A_j \end{pmatrix}\}$$

(A6) Decomposability:

$$\begin{pmatrix} p & 1-p \\ A_i & \begin{pmatrix} q & 1-q \\ A_j & A_h \end{pmatrix} \end{pmatrix} \sim$$
$$\begin{pmatrix} p & (1-p)q & (1-p)(1-q) \\ A_i & A_j & A_h \end{pmatrix}$$

Given the above axioms, there exists a real valued function U which verifies

$$\begin{aligned} U(A_i) > U(A_j) &\leftrightarrow A_i \succ A_j \\ U(A_i) = U(A_j) &\leftrightarrow A_i \sim A_j \end{aligned}$$

Such a function, defined over the set of states and lotteries, can be supposed to be normalized, that is to say, to have values between 0 (minimal utility) and 1 (maximal utility). A possible way to define a utility function is the following:

- let A be a state of maximal utility and A' be a state of minimal utility;
- define $U(A) = 1$ and $U(A') = 0$;
- for any other state A_i, in view of axiom (A3), there exists a probability value p such that

$$A_i \sim \begin{pmatrix} p & 1-p \\ A & A' \end{pmatrix}$$

Put then $U(A_i) = p$.

3. Fuzzy preferences

Fuzzy preference relations were introduced by Zadeh [14] in order to capture degrees of preferences. Two different alternatives may be considered to be better than a third one, but one preference may not be as intense as the other. In this way, given a finite set of alternatives or states X, a fuzzy binary preference relation is defined as a fuzzy set over all pairs of the cartesian product $X \times X$, so that its membership function, $\mu : X \times X \rightarrow [0,1]$, associates to each pair (x, y) the strength or intensity of preference $\mu(x,y)$ between x and y, measured in the unit interval. More intuitively, $\mu(x,y)$ gives the degree to which x is not considered to be worse than y.

In some cases, we can also assume that the fuzzy binary preference relations μ are complete, where completeness is defined as

$$\mu(x,y) + \mu(y,x) \geq 1 \ \forall x, y \in X.$$

Completeness is required in order to assure that the set of states is feasible and comprehensive (see [10], for example, for an axiomatic discussion). Then, the values

$$\begin{aligned} \mu_I(x,y) &= \mu(x,y) + \mu(y,x) - 1 \\ \mu_B(x,y) &= \mu(x,y) - \mu_I(x,y) \\ \mu_W(x,y) &= \mu(y,x) - \mu_I(x,y) \end{aligned}$$

can be understood, respectively, as the degree to which the two alternatives are indifferent (xIy), the degree of strict preference of x over y (xBy, *x is better than y*) and the degree of strict preference of y over x (xWy, *x is worse than y*). In this way, preference between two alternatives x and y will be explained by means of these three intensity values (xBy, xIy, xWy), each one associated to one possible crisp relation between both alternatives in such a way that

$$\mu_B(x,y) + \mu_I(x,y) + \mu_W(x,y) = 1 \ \forall x, y \in X.$$

4. Fuzzy rationality measures

Given now a complete (or the completion of a non complete one) fuzzy preference relation μ, we must assign to μ a degree or rationality. In the context of fuzzy binary preference relations, a standard assumption to characterize consistency or rationality is max-min transitivity (see [14] and [8, 15]).

General fuzzy rationality measures, that is to say functions ρ which map fuzzy preference relations into the interval $[0,1]$ and which allow a fuzzy classification have been introduced in [5] and characterized in depth in [6].

5. The axioms of fuzzy utility theory

Let $A_1, \ldots, A_n$ be a set of basic alternatives or states. Below is the list of the extension of the utility theory axioms as proposed in [7]. In particular, for what concerns the first axiom we notice that since we want to deal with rational agents we suppose that the preference relation μ defined over the set of basic alternatives must verify $\rho(\mu) > 0$, for some fuzzy rationality measure ρ. Such a property is obviously the extension to the fuzzy case of the axioms (A1) and (A2) of utility theory, above described.

(FA1) Rationality:

$$\rho(\mu) > 0$$

(FA2) Extension: for any probability value p, and basic alternatives A_i, A_j, A_h

$$\min(\mu(A_h, A_i), \mu(A_h, A_j)) \leq \mu(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix})$$

and

$$\mu(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}) \leq \max(\mu(A_h, A_i), \mu(A_h, A_j))$$

Analogously,

$$\min(\mu(A_i, A_h), \mu(A_j, A_h)) \leq \mu(\begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}, A_h)$$

and

$$\mu(\begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}, A_h) \leq \max(\mu(A_i, A_h), \mu(A_j, A_h))$$

In particular, we have

- for $\mu(A_h, A_i) = 1$ and for any p,

$$\mu(A_h, A_j) \leq \mu(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}) \leq 1$$

- for $\mu(A_i, A_h) = 1$ and for any p,

$$\mu(A_j, A_h) \leq \mu(\begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}, A_h) \leq 1$$

As a consequence, if $\mu(A_h, A_i) = \mu(A_i, A_h) = 1$

$$\mu_I(A_h, A_j) \leq \mu_I(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}).$$

In view of axiom (FA2) we can propose an analytical definition of the fuzzy preference relation, to be axiomatically justified.

DEFINITION 1 (F-def) *For any probability value p, and alternatives A_i, A_j, A_h*

$$\mu(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}) = p\mu(A_h, A_i) + (1-p)\mu(A_h, A_j)$$

and

$$\mu(\begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}, A_h) = p\mu(A_i, A_h) + (1-p)\mu(A_j, A_h).$$

Moreover, given any $k \geq 2$, $X_1, \ldots, X_k$ alternatives and $p_1, \ldots, p_k$ such that $\sum_{i=1}^k p_i = 1$ we extend μ as follows

- *let* $B = \begin{pmatrix} p_1 & \cdots & p_k \\ X_1 & \cdots & X_k \end{pmatrix}$,
- *given any alternative A and probability value q let*

$$C = \begin{pmatrix} q & (1-q)p_1 & \cdots & (1-q)p_k \\ A & X_1 & \cdots & X_k \end{pmatrix}$$

then

$$\mu\left(\begin{pmatrix} q & 1-q \\ A & B \end{pmatrix}, C\right) = \mu\left(C, \begin{pmatrix} q & 1-q \\ A & B \end{pmatrix}\right) = 1.$$

From (F-def) it follows that

$$\mu_f\left(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}\right) = p\mu_f(A_i, A_h) + (1-p)\mu_f(A_j, A_h)$$

It also follows that the decomposability axiom is properly extended.

Notice that in case μ is crisp, and $A_i \succ A_h \succ A_j$, then we may suppose that $\mu\left(A_h, \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}\right) = 1$ if $p \leq 1/2$ and 0 otherwise. Analogously, we may suppose that $\mu\left(\begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}, A_h\right) = 1$ if $p \geq 1/2$ and 0 otherwise. That is to say, we put 1/2 as a threshold and we increase μ to 1 if it is at least 1/2, whereas we decrease to 0 if it less than 1/2. In this case, for $p = 1/2$, we obtain $A_h \sim \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}$, which, without differences in intensity of preferences, seems to be the only appropriate understanding of axiom (A3) of continuity. We can conclude then that axiom (FA2) as extended by (F-def) is also a fuzzy extension of axiom (A3).

Let now $X = \begin{pmatrix} p & 1-p \\ A_i & A_h \end{pmatrix}$ and $Y = \begin{pmatrix} p & 1-p \\ A_j & A_h \end{pmatrix}$. If $\mu_f(A_i, A_j) = 1$ can we automatically claim that $\mu_f(X, Y) = 1$? The answer is no! Indeed, we must take into account the preference values which relate A_i and A_j to A_h. Following (F-def) we have

- for $p = 1$, $\mu_f(X, Y) = \mu_f(A_i, A_j)$
- for $p = 0$, $\mu_f(X, Y) = 1$

In general, we have

$$\mu_f(X, Y) = p^2\mu_f(A_i, A_j) + p(1-p)(\mu_f(A_i, A_h) + \mu_f(A_j, A_h)) + (1-p)^2$$

The monotonicity condition is implied by (F-def). Specifically, suppose that $\mu(A_i, A_j) > \mu(A_j, A_i)$ and let $X = \begin{pmatrix} p & 1-p \\ A_i & A_j \end{pmatrix}$, $Y = \begin{pmatrix} q & 1-q \\ A_i & A_j \end{pmatrix}$ and $\mu_{ij} = \mu(A_i, A_j)$ then for all probability values p, q we have

$$\mu(X, Y) = pq + (1-p)q\mu_{ji} + p(1-q)\mu_{ij} + (1-p)(1-q)$$

and

$$\mu(Y, X) = pq + (1-q)p\mu_{ji} + q(1-p)\mu_{ij} + (1-p)(1-q)$$

Thus

$$\mu(X, Y) - \mu(Y, X) = \alpha(\mu_{ij} - \mu_{ji})$$

where $\alpha = p(1-q) - (1-p)q$ and it is clear that $\alpha \geq 0$ if and only if $p \geq q$.

6. Defining a utility function

We use the 5-th and last axiom of fuzzy rationality measures which partitions the set of rationality measures into three sets: normal, pessimistic and optimistic.

(R5) Regularity: Given $\mu : X \times X \to [0, 1]$ defined over X, $(\bar{x}, \bar{y})$ arbitrary pair of alternatives, let $P_\mu(\bar{x}, \bar{y})$ be the collection of pair of real numbers (a, b) such that

(i) $0 \leq \mu(\bar{x}, \bar{y}) + a \leq 1$

(ii) $0 \leq \mu(\bar{y}, \bar{x}) + b \leq 1$

(iii) $\mu(\bar{x}, \bar{y}) + \mu(\bar{y}, \bar{x}) + a + b \geq 1$

For every pair $(a, b) \in P_\mu(\bar{x}, \bar{y})$ we will denote by

$$\Delta_\mu((\bar{x}, \bar{y}), (a, b))$$

the fuzzy preference relation defined on the pair (x, y) as follows

$$\begin{cases} \mu(\bar{x}, \bar{y}) + a & \text{if } (x, y) = (\bar{x}, \bar{y}) \\ \mu(\bar{y}, \bar{x}) + b & \text{if } (x, y) = (\bar{y}, \bar{x}) \\ \mu(x, y) & \text{otherwise} \end{cases}.$$

Let ρ be a fuzzy rationality measure, then

(1) ρ is normal if

$$\rho(\Delta_\mu((\bar{x}, \bar{y}), (a, 0)))$$
$$\rho(\Delta_\mu((\bar{x}, \bar{y}), (a, -a)))$$
$$\rho(\Delta_\mu((\bar{x}, \bar{y}), (a, a)))$$

are monotone functions of a.

(2) ρ is pessimistic if there exists a value $a \in [0, 1]$ such that

(2.1) $\rho(\Delta_\mu((\bar{x}, \bar{y}), (b, 0))) \geq \rho(\Delta_\mu((\bar{x}, \bar{y}), (c, 0)))$

(2.2) $\rho(\Delta_\mu((\bar{x}, \bar{y}), (b, -b))) \geq \rho(\Delta_\mu((\bar{x}, \bar{y}), (c, -c)))$

(2.3) $\rho(\Delta_\mu((\bar{x}, \bar{y}), (b, b))) \geq \rho(\Delta_\mu((\bar{x}, \bar{y}), (c, c)))$

for all b, c such that either $a > b > c$ or $a < b < c$.

(3) ρ is optimistic if there exists a value $a \in [0,1]$ such that

(3.1) $\rho(\Delta_\mu((\bar{x},\bar{y}),(b,0))) \le \rho(\Delta_\mu((\bar{x},\bar{y}),(c,0)))$

(3.2)

$$\rho(\Delta_\mu((\bar{x},\bar{y}),(b,-b))) \le \rho(\Delta_\mu((\bar{x},\bar{y}),(c,-c)))$$

(3.3) $\rho(\Delta_\mu((\bar{x},\bar{y}),(b,b))) \le \rho(\Delta_\mu((\bar{x},\bar{y}),(c,c)))$

for all b,c such that either $a > b > c$ or $a < b < c$.

We have the following

THEOREM 1 (Boosting Theorem) *By using the regularity axiom we can obtain from μ a fuzzy preference relation μ^* such that*

- $\rho(\mu^*) \ge \rho(\mu)$
- *for every pair of alternatives (x,y) either*
  - $\max\{\mu^*(x,y),\mu^*(y,x)\} = 1$, *or*
  - $\min\{\mu^*(x,y),\mu^*(y,x)\} = 0$.

Then we can partition X in several subsets $X_1,\dots,X_k$ such that for every $x \in X_i$ and $y \in X_j$ if $i < j$ then either $\mu(x,y) = 1$ and $\mu(y,x) < 1$, or $\mu(x,y) > 0$ and $\mu(y,x) = 0$.

Having *linearized* the alternatives we can act as in the classical crisp case.

7. Final comments

Following a given formalization of a possible fuzzy extension of the classical axioms of utility theory we provide in this paper a way to construct a utility function by, algorithmically boosting the fuzzy preference, without loss of rationality.

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